

A Paley-Wiener Like Theorem for Nilpotent Lie Groups

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February 7, 2008

Dedicated to the memory of G. Polya

Abstract

A version of Paley-Wiener like theorem for connected, simply connected nilpotent Lie groups is proven.

1 Introduction

In the paper [1] a version of Paley-Wiener theorem for two- and three-step nilpotent Lie group was proven. To this end R. PARK developed a subtle technique for analysis on nilpotent Lie groups. It seems that the technique is of separate interest and could be used to study other problems.

However, it is possible to give a shorter proof of a more general theorem based on the completely standard results about nilpotent Lie groups. This is the goal of the present paper. We will prove the result of PARK for all connected, simply connected (exponential) nilpotent Lie groups, but even this is not last level of generality—see Remark 3.2. It turns to be that the theorem in such a form is a direct consequence of the one-dimensional Paley-Wiener theorem. It seems that this is another example of *inventor paradox* [2].

In Section 2 we give standard facts about nilpotent Lie groups, which will be used in Section 3 to prove a version of Paley-Wiener theorem.

2 Preliminaries

The following information about nilpotent Lie group could be found in original papers [3, 4] or in monographs [5] and [6, Chap. 6]. It should be noted that the method of orbits and the induced representations technique are closely connected for nilpotent Lie groups.

Let \mathfrak{g} be a nilpotent Lie algebra of the dimension n , let $G = \exp(\mathfrak{g})$ be the connected, simply connected nilpotent Lie group corresponding to it. We will identify \mathfrak{g} and G via the exponential map and will consider both of them coinciding with \mathbb{R}^n (as vector spaces and a C^∞ -manifolds respectively). We also will denote by a common letter a representation π of G and its derived representation of \mathfrak{g} . All irreducible unitary representations could be constructed by the inductive procedure [3], [6, Chap. 6]. We have

- 2.1. The group G is unimodular, the two-sided Haar measure $d\mu$ on G coincides with the Lebesgue measure on \mathbb{R}^n . Any Lie subgroup H is also nilpotent and homeomorphic to \mathbb{R}^m for some $m \leq n$. The homogeneous space $X = H \backslash G$ is homeomorphic to \mathbb{R}^{n-m} . Invariant measures on H and X coincide again with the Lebesgue measures on \mathbb{R}^m and \mathbb{R}^{n-m} correspondingly.

- 2.2. The unitary dual \widehat{G} could be parametrized by orbits of co-adjoint representation in \mathfrak{g}' . The support \widetilde{G} of the Plancherel measure $d\nu$ in \widehat{G} corresponds to orbits of maximal dimensionality and parametrized by $\mathbb{R}^k \subset \mathfrak{g}'$, where $\dim \mathfrak{g} = n$ and $n - k$ is the maximal dimensionality of orbits. Moreover [3, § 7.4], the Plancherel measure is equivalent to the Lebesgue measure on $\widetilde{G} \cong \mathbb{R}^k$:

$$d\mu = R(\lambda) d\lambda_1 \dots d\lambda_k. \quad (2.1)$$

- 2.3. Every unitary irreducible representation π of G is induced by a one-dimensional representation π_0 of a subgroup $H \subset G$ [3, Theorem 5.1]. The last representation has a form

$$\pi_0(\exp A) = \exp(i\langle l, A \rangle), \quad A \in \mathfrak{h} \quad (2.2)$$

for some $l \in \mathfrak{g}'$.

- 2.4. According to the general scheme of induced representation (see [5, § 13.2], [6, Chap. 5]) the representation could be realized as follows. Let $L_2(G, H, \pi_0)$ be the space of functions on G with the property

$$F(hg) = \pi_0(h)F(g), \quad h \in H, \quad g \in G.$$

Then π is equivalent to the representation on $L(G, H, \pi_0)$ by the shift:

$$[\pi(g)F](g_1) = F(g_1g). \quad (2.3)$$

There is an alternative representation. Let $X = H \backslash G$ be a homogeneous space, $s : X \rightarrow G$ be a measured mapping, such that $s(Hg) \in Hg$. Define an isomorphism $L_2(G, H, \pi) \rightarrow L_2(X)$ as

$$f(x) = F(s(x)), \quad F(hs(x)) = \pi_0(h)f(x), \quad x \in X, \quad h \in H. \quad (2.4)$$

Then

$$[\pi(g)f](x) = A(g, x)f(xg), \quad (2.5)$$

where

$$A(g, x) = \pi_0(h), \quad hs(xg) = s(x)g.$$

3 A Paley-Wiener Theorem

Theorem 3.1 *Let $G = \exp(\mathfrak{g})$ be a connected, simply connected nilpotent Lie group. Let $\phi(g)$ be a function from $L_\infty(G)$ with a compact support. Then if $\widehat{\phi}(\pi) = 0$ on a subset E of \widehat{G} of a positive Plancherel measure then $\phi(g) = 0$ almost everywhere on G .*

PROOF. We start from the change of variables in the formula defines the non-commutative Fourier transform $\widehat{\phi}(\pi)$ of ϕ in the form (2.3) of induced representations:

$$\begin{aligned} [\widehat{\phi}(\pi)F](g_1) &= \int_G \phi(g)F(g_1g') d\mu(g') \\ &= \int_G \phi(g_1^{-1}g)F(g) d\mu(g). \end{aligned}$$

We rewrite the last line for the representation (2.5) using connection (2.4):

$$\begin{aligned} [\widehat{\phi}(\pi)f](x_1) &= \int_G \phi([s(x_1)]^{-1}hs(x))\pi_0(h)f(x) d\mu(hs(x)) \\ &= \int_X \int_H \phi([s(x_1)]^{-1}hs(x))\pi_0(h)f(x) dh dx \\ &= \int_X \int_H \phi([s(x_1)]^{-1}hs(x))\pi_0(h) dh f(x) dx \\ &= \int_X \left(\int_H \phi([s(x_1)]^{-1}hs(x)) \exp(i\langle l, \log h \rangle) dh \right) f(x) dx \end{aligned} \quad (3.1)$$

(We substitute in (3.2) the form (2.2) of representation π_0 .) Thus $\widehat{\phi}(\pi)$ is an integral operator $K(l) : L_2(X) \rightarrow L_2(X)$ of the form

$$[K(l)f](x_1) = \int_X K(l, x_1, x) f(x) dx$$

with the kernel $K(l, x_1, x)$ defined via the usual Fourier transform with respect to variables $h \rightarrow l$:

$$K(l, x_1, x) = \int_H \phi([s(x_1)]^{-1}hs(x)) \exp(i\langle l, \log h \rangle) dh.$$

If the operator $K(l)$ is equal to 0 for a fixed l , then the kernel $K(l, x_1, x)$ should be equal to zero almost everywhere for $(x_1, x) \in X \times X$. But if the

last statement fulfils for all $l \in E$, where $E \subset \mathbb{R}^k$ has a non-zero measure, then $\phi(g) = 0$ a.e. by the standard Paley-Wiener theorem. \square

REMARK 3.2 To prove the theorem we have used only properties 2.1–2.4. It is well known [3], [6, Chap. 6] that these properties are not too specific and shared also, for example, by type I solvable Lie group. So it is naturally to assume that our theorem and its proof could be also transformed to this more general case. Reader could find a version of our transformations (3.1)–(3.2) for the group $SL(2, \mathbb{R})$ in [7, § III.4].

We are tempted to conclude the paper by the following

Problem 3.3 Let G be a nilpotent (or solvable) Lie group. Find conditions for the function ϕ , which guarantees that its Fourier transform $\widehat{\phi}(\pi)$ is invertible almost everywhere in the Plancherel measure on \widehat{G} .

4 Acknowledgments

The author was partially supported by the INTAS grant 93–0322. During the preparation of the paper in fall of 1995 the author enjoyed the hospitality of Universiteit Gent, Vakgroep Wiskundige Analyse, Belgium. The author is also very grateful to D.S. Kalyuzhny and the unknown referee of *Journal of Functional Analysis* for useful comments.

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